

# BOGOLYUBOV INEQUALITY FOR THE GROUND STATE AND ITS APPLICATION TO INTERACTING ROTOR SYSTEMS.

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**ABSTRACT.** We have formulated and proved the Bogolyubov inequality for operators at zero temperature. So far this inequality has been known for matrices, and we were able to extend it to certain class of operators. We have also applied this inequality to the system of interacting rotors. We have shown that if: *i)* the dimension of the lattice is 1 or 2, *ii)* the interaction decreases sufficiently fast with a distance, and *iii)* there is an energy gap over the ground state, then the spontaneous magnetization in the ground state is zero, i.e. there is no LRO in the system. We present also heuristic arguments (of perturbation-theoretic nature) suggesting that one- and two-dimensional system of interacting rotors has the energy gap independent of the system size if the interaction is sufficiently small.

*Keywords:* Statistical mechanics; operator inequalities; Bogolyubov inequality; Mermin-Wagner theorem

## 1. INTRODUCTION

The *Mermin – Wagner theorem* [1] is one of the most important results, concerning phase transitions or their absence in systems described by statistical mechanics. In short, it claims that *in one- and two-dimensional quantum lattice systems, possessing continuous symmetry, and with short-range interactions, there is no Long-Range Order (LRO) in non-zero temperature.*

The Mermin-Wagner theorem (M-W theorem) has been generalized in many directions. Some of the most important results are: no LRO in boson systems [2]; extension of the theorem to classical models [3]; spatial fall-off of correlation functions [4]; lack of magnetic and superconducting orderings in itinerant fermion systems (Hubbard-like models) [5], [6]; quantum rotor systems [7]; no LRO in spin systems with lack of translational symmetry [8]. For a review, see [9].

The original M-W theorem deals only with positive temperature but, under certain additional assumptions, it has also been generalized to the *zero temperature* [12][13]. It turns out that the ground state of one- or two-dimensional systems with continuous symmetry group can be ordered in some cases (XY or Heisenberg models on square lattice – see [10]), whereas other models possess disordered ground state ('Resonating Valence Bond models' – see [11]). One possible source of a disorder at zero temperature is *an energy gap* between the ground state and excited states: if a given system is gapped, then it is disordered also in the ground state. This has been shown for the lattice spin systems [12] and itinerant fermion systems [13], [14], [15].

The M-W theorem has been proved with the use of *Bogolyubov inequality* [17]. (Later on, there appeared also another techniques of proving lack of LRO in one- and two-dimensional systems with continuous symmetry group for positive temperatures; for an excellent presentation, see [18]).

The Bogolyubov inequality is also the basic technical tool in extensions of M-W theorems. An important aspect here is the dimension of the Hilbert space on a single site. If this dimension is finite, then the Bogolyubov inequality is a matrix inequality and its proof is relatively easy. However, when this dimension is infinite, then certain operator-theoretic considerations enter the game. Such an infinite-dimensional version of the Bogolyubov inequality (for positive temperature) has been proved in [19] and applied in an analogon of the M-W theorem for interacting rotors in [7].

However – to our best knowledge – the question of infinite dimensional version of the Bogolyubov inequality for *zero temperature* is still open. It seems that the same concerns the zero-temperature M-W theorem for systems with infinite dimensional Hilbert space. This opportunity motivated us to undertake efforts on these areas. In the present paper, we describe the results obtained. They can be summarized as follows:

- i) We have formulated and proved an infinite-dimensional version of the Bogolyubov inequality for zero temperature.
- ii) We have applied it to the system of interacting rotors and have shown that if there exists an energy gap over the ground state, then there is no LRO at zero temperature.

We also present (non-rigorous) arguments for existence of the gap if the interaction between rotors is sufficiently small.

The organization of the paper is as follows. In the next section we formulate and prove the zero-temperature version of the Bogolyubov inequality for certain class of operators. In the third section, we define Hamiltonians for interacting rotor systems, the magnetization as the measure of ordering, and prove the zero-temperature Mermin-Wagner theorem for these systems. We consider the planar and spherical rotors. We present also arguments for existence of energy gap in the weakly interacting rotor systems; they are based on perturbation theory. The fourth section is devoted to the summary and description of some open problems. The Appendix contains conventions and formulas for Legendre Polynomials and spherical harmonics we use.

## 2. ZERO TEMPERATURE BOGOLIUBOV INEQUALITY

**2.1. Finite dimensional Bogolyubov inequalities.** We begin by recalling some well known facts concerning the original Bogolyubov inequality [1]. Let  $\mathcal{H}$  be a *finite dimensional* Hilbert space and  $H$  a selfadjoint operator on  $\mathcal{H}$  (the Hamiltonian of the system). Define the function [1]  $(\cdot, \cdot) : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ :

$$(1) \quad (A, B) := \frac{1}{\text{Tr}(e^{-\beta H})} \sum'_{i,j} (j | A^\dagger i) (i | B j) \frac{e^{-\beta E_i} - e^{-\beta E_j}}{E_j - E_i},$$

where the summation is performed over all eigenstates of  $H$  and  $E_i$  is an eigenenergy corresponding to the  $i$ -th eigenstate. The prime at the sum sign denote that the summation is performed excluding pairs with the same energy.

**Remark.** The function (1) is closely related to the *Duhamel function*, see [18], [20]

One checks that  $(A, B)$  is a hermitean, sesquilinear, non-negative form on  $\mathcal{B}(\mathcal{H})$ . Therefore the Schwarz inequality holds:

$$|(A, B)|^2 \leq (A, A)(B, B).$$

Using this formula one derives the *Bogolyubov inequality*:

$$(2) \quad |\langle [A, B] \rangle|^2 \leq \frac{\beta}{2} \langle [[B, H], B^\dagger] \rangle \langle A^\dagger A + A A^\dagger \rangle$$

where  $\langle \cdot \rangle$  denote standard thermal average. The definition of  $(A, B)$  and the Bogolyubov inequality are meaningful for positive temperatures. However, one can modify them to the situation of zero temperature ( $\beta = \infty$ ) [13]. To this aim let us define:

$$(3) \quad (A, B) := \frac{1}{Z} \sum_m \sum_\alpha \frac{(\alpha | A^\dagger m)(m | B \alpha) + (\alpha | B m)(m | A^\dagger \alpha)}{E_m - E_0}$$

where  $\alpha$  is an index for ground states,  $Z = \sum_\alpha (\alpha | \alpha)$ ,  $E_0$  is the ground state energy and  $m$  refers to excited states with eigenenergy  $E_m$ . As before, (3) defines a hermitean, sesquilinear, non-negative form on  $\mathcal{B}(\mathcal{H})$ . Using the Schwarz inequality one derives the *zero-temperature version of the Bogolyubov inequality*:

$$(4) \quad |\langle [A, B] \rangle_0|^2 \leq \frac{1}{\Delta} \langle [B, [H, B^\dagger]] \rangle_0 \langle A^\dagger A + A A^\dagger \rangle_0$$

where the average is taken over the ground states of the system and  $\Delta := E_1 - E_0$ . (See also [12], [15] for related considerations.) Our goal now is to extend the definition of  $(A, B)$  and the Bogolyubov inequality for  $A, B$  being operators on an infinite dimensional Hilbert space.

**2.2. Infinite dimensional, zero temperature Bogolyubov inequality.** Let  $H$  be a selfadjoint, bounded from below operator with compact resolvent acting on a separable Hilbert space  $\mathcal{H}$  and let  $H = \sum_{n=0}^{\infty} E_n P_n$  be its spectral decomposition. Thus we have:

$$(5) \quad \begin{aligned} E_0 < E_1 < \dots < E_n < \dots, \quad \lim_{n \rightarrow \infty} E_n = \infty \\ P_n^* P_m &= \delta_{nm} P_n, \quad \sum_n P_n = I, \quad N_n := \dim P_n(\mathcal{H}) < \infty. \end{aligned}$$

For further considerations we shall fix an  $H$ -invariant subspace  $\mathcal{S}_H \subset \mathcal{H}$  such that  $P_n(\mathcal{H}) \subset \mathcal{S}_H$  for any  $n = 0, 1, \dots$ . Clearly  $\mathcal{S}_H$  is dense in  $\mathcal{H}$ . Therefore any linear (possibly unbounded) operator  $C : \mathcal{S}_H \rightarrow \mathcal{H}$  ( $C \in \mathcal{L}(\mathcal{S}_H, \mathcal{H})$ ) has a well defined adjoint operator.

In what follows we shall consider the following set of operators

$$(6) \quad Op(\mathcal{S}_H) := \{C \in \mathcal{L}(\mathcal{S}_H, \mathcal{H}) : \mathcal{S}_H \subset D(C^*)\}$$

Notice that for any  $C \in Op(\mathcal{S}_H)$  the operator  $C^*$  is densely defined, so  $C$  is closeable and  $\overline{C} = C^{**}$ . Let  $C^\dagger := C^*|_{\mathcal{S}_H}$ . Moreover  $C^\dagger \in Op(\mathcal{S}_H)$ ; in fact, since  $C^\dagger \subset C^*$ , we have  $\overline{C} = C^{**} \subset (C^\dagger)^*$  i.e.  $C \subset \overline{C} \subset (C^\dagger)^*$ . Now,  $(C^\dagger)^\dagger = (C^\dagger)^*|_{\mathcal{S}_H} = C$  and the map  $Op(\mathcal{S}_H) \ni C \mapsto C^\dagger \in Op(\mathcal{S}_H)$  is an *antilinear involution*.

A *bounded* operator  $A$  acting on  $\mathcal{H}$  defines an element of  $Op(\mathcal{S}_H)$  by restriction to  $\mathcal{S}_H$ . On the other hand  $A = \overline{A|_{\mathcal{S}_H}}$ . In this sense we shall write  $\mathcal{B}(\mathcal{H}) \subset Op(\mathcal{S}_H)$ . Clearly  $\mathcal{S}_H$  is an essential domain for  $H$  and for  $H_0 := H|_{\mathcal{S}_H}$  we have  $H_0 \in Op(\mathcal{S}_H)$  and  $H_0^\dagger = H_0$ . Notice that  $Op(\mathcal{S}_H)$  is a linear space. For any  $C \in Op(\mathcal{S}_H)$  and any  $n = 0, 1, \dots$  operators  $CP_n$  and  $C^*P_n = C^\dagger P_n$  are well defined and have finite dimensional rank.

**Lemma 2.1.** *a) Let  $A \in \mathcal{B}(\mathcal{H})$  and  $A(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then  $CA, A^*C \in Op(\mathcal{S}_H)$  for any  $C \in Op(\mathcal{S}_H)$ .  
b) Let  $C, B \in Op(\mathcal{S}_H)$  such that  $C(\mathcal{S}_H), B(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then  $C^\dagger B \in Op(\mathcal{S}_H)$ .  
c) Let  $C \in Op(\mathcal{S}_H)$  such that  $C(\mathcal{S}_H), C^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then  $[C^\dagger, H_0], [C, H_0] \in Op(\mathcal{S}_H)$ .*

*Proof:* Ad a) Clearly  $CA, A^*C \in \mathcal{L}(\mathcal{S}_H, \mathcal{H})$ . Moreover  $\mathcal{S}_H \subset D((CA)^*)$  due to the inclusion  $(CA)^* \supset A^*C^*$ . Similarly,  $(A^*C)^* \supset C^*A$  and  $\mathcal{S}_H$  is  $A$  invariant, therefore  $\mathcal{S}_H \subset D((A^*C)^*)$ .

Ad b) By  $B$  invariance of  $\mathcal{S}_H$ :  $C^\dagger B : \mathcal{S}_H \rightarrow \mathcal{H}$ . Since  $(C^\dagger B)^* \supset B^*(C^\dagger)^* \supset B^*C$  and  $\mathcal{S}_H$  is  $C$ -invariant, the domain of  $B^*C$  contains  $\mathcal{S}_H$ , so  $C^\dagger B \in Op(\mathcal{S}_H)$ .

Ad c) Since  $[C^\dagger, H_0] = C^\dagger H_0 - H_0 C^\dagger \in \mathcal{L}(\mathcal{S}_H, \mathcal{H})$  this assertion follows from b) and the fact that  $Op(\mathcal{S}_H)$  is a linear space. ■

Let us define a family of sesquilinear forms  $\rho_{nk}$  on  $Op(\mathcal{S}_H)$  by:

$$(7) \quad \rho_{nk}(C, B) := \text{Tr}(P_k C^* P_n B P_k) = \sum_{j=1}^{N_k} (C e_j | P_n B e_j), \quad n, k = 0, 1, 2, \dots$$

where  $(e_j)$  is an o.n basis in  $P_k(\mathcal{H})$  (and  $N_k := \dim P_k(\mathcal{H})$ ).

**Lemma 2.2.** (1)  $\rho_{nk}$  are non-negative, hermitean forms on  $Op(\mathcal{S}_H)$ ,  
(2)  $\rho_{nk}(C, B) = \rho_{kn}(B^\dagger, C^\dagger)$  for any  $C, B \in Op(\mathcal{S}_H)$   
(3) Let  $C, B \in Op(\mathcal{S}_H)$  such that  $C(\mathcal{S}_H), C^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then

$$\rho_{nk}(B, [C, H_0]) = (E_k - E_n) \rho_{nk}(B, C).$$

(4) For any  $C, B \in Op(\mathcal{S}_H)$  the series  $\sum_n \rho_{nk}(C, B)$  is absolutely convergent and

$$\sum_n \rho_{nk}(C, B) = \sum_{j=1}^{N_k} (C e_j | B e_j),$$

where  $(e_j)$  is o.n. basis in  $P_k(\mathcal{H})$ .

*Proof:* Statement (1) is a direct consequence of the definition;. The statement (2) follows from calculation:

$$\begin{aligned} \rho_{nk}(C, B) &= \text{Tr}(P_k C^* P_n B P_k) = \text{Tr}(P_n B P_k C^* P_n) = \text{Tr}(P_n B P_k C^\dagger P_n) = \text{Tr}(P_n (B^\dagger)^* P_k C^\dagger P_n) \\ &= \rho_{kn}(B^\dagger, C^\dagger). \end{aligned}$$

To prove the statement (3) let us observe that by lemma 2.1(c)  $[C, H_0] \in Op(\mathcal{S}_H)$ . Since  $H_0 P_k = E_k P_k$  and  $P_n H_0 C P_k = E_n P_n C P_k$  we have:

$$\begin{aligned} \rho_{nk}(B, [C, H_0]) &= \text{Tr}(P_k B^* P_n (C H_0 - H_0 C) P_k) = \text{Tr}(P_k B^* P_n C H_0 P_k - P_k B^* P_n H_0 C P_k) = \\ &= \text{Tr}(E_k P_k B^* P_n C P_k - E_n P_k B^* P_n C P_k) = (E_k - E_n) \text{Tr}(P_k B^* P_n C P_k) \\ &= (E_k - E_n) \rho_{nk}(B, C). \end{aligned}$$

Let us prove (4). For any  $C \in Op(\mathcal{S}_H)$  the positive series  $\sum_n \rho_{nk}(C, C)$  is convergent:

$$\begin{aligned} \infty &> \sum_{j=1}^{N_k} (C e_j | C e_j) = \sum_{j=1}^{N_k} \sum_n (C e_j | P_n C e_j) = \sum_n \sum_{j=1}^{N_k} (C e_j | P_n C e_j) = \\ &= \sum_n \text{Tr}(P_k C^* P_n C P_k) = \sum_n \rho_{nk}(C, C). \end{aligned}$$

Now the assertion follows from the polarization formula for  $\rho_{nk}(C, B)$ . ■

Let us define:

$$(8) \quad Op(\mathcal{S}_H) \times Op(\mathcal{S}_H) \ni (B, C) \mapsto (B, C)_0 := \sum_{n>0} \frac{1}{E_n - E_0} (\rho_{n0} + \rho_{0n})(B, C)$$

Notice that the sequence  $a_n := \frac{1}{E_n - E_0}$ ,  $n \neq 0$  is bounded so by (2) and (4) of the lemma 2.2 the series in (8) is absolutely convergent and the definition makes sense.

**Proposition 2.3.** *The form  $(\cdot, \cdot)_0$  is sesquilinear, hermitean and non-negative.*

Moreover for any  $B, C \in Op(\mathcal{S}_H)$ :

- (1)  $(B, C)_0 = (C^\dagger, B^\dagger)_0$ ;
- (2)  $(B, [C, H_0])_0 = \text{Tr}(P_0 [C, B^\dagger] P_0)$ , whenever  $B^\dagger(\mathcal{S}_H), C^\dagger(\mathcal{S}_H), C(\mathcal{S}_H) \subset \mathcal{S}_H$ .

*Proof:* The form  $(\cdot, \cdot)_0$  is a sum of hermitean, non-negative forms. The formula (1) follows from statement (2) of the Lemma 2.2. By the item (3) Lemma 2.2:

$$\frac{1}{E_n - E_0} (\rho_{n0} + \rho_{0n})(B, [C, H_0]) = -\rho_{n0}(B, C) + \rho_{0n}(B, C) = \rho_{n0}(C^\dagger, B^\dagger) - \rho_{n0}(B, C).$$

Now, using the item (4) of the same lemma:

$$\begin{aligned} \sum_n [\rho_{n0}(C^\dagger, B^\dagger) - \rho_{n0}(B, C)] &= \sum_{j=1}^{N_0} [(C^\dagger e_j | B^\dagger e_j) - (B e_j | C e_j)] = \sum_{j=1}^{N_0} [(e_j | C B^\dagger e_j) - (e_j | B^\dagger C e_j)] = \\ &= \text{Tr}(P_0 [C, B^\dagger] P_0), \end{aligned}$$

where the last but one equality follows from the inclusion  $B^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ . ■

**Corollary 2.4.** *Let  $B, C \in Op(\mathcal{S}_H)$  such that  $C(\mathcal{S}_H), C^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then*

$$|(B, [C^\dagger, H_0])_0|^2 \leq (B, B)_0 ([C^\dagger, H_0], [C^\dagger, H_0])_0$$

*Proof:* This is the Schwarz inequality for  $(\cdot, \cdot)_0$  and operators  $B$  and  $[C^\dagger, H_0]$  (cf. statement (c) of Lemma 2.1). ■

Let us observe that  $[C^\dagger, H_0]^\dagger = [H_0, C]$ , so by (2) of Prop.2.3 we have

$$([C^\dagger, H_0], [C^\dagger, H_0])_0 = \text{Tr}(P_0[C^\dagger, [H_0, C]]P_0)$$

Now Corollary 2.4 reads:

$$|(B, [C^\dagger, H_0])_0|^2 \leq (B, B)_0 \text{Tr}(P_0[C^\dagger, [H_0, C]]P_0)$$

If in addition  $B^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ , by the same argument as above, we obtain:

$$(B, [C^\dagger, H_0])_0 = \text{Tr}(P_0[C^\dagger, B^\dagger]P_0).$$

This way we have proved:

**Proposition 2.5.** *Let  $B, C \in Op(\mathcal{S}_H)$  be such that  $B^\dagger(\mathcal{S}_H), C(\mathcal{S}_H), C^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then*

$$(9) \quad |\text{Tr}(P_0[C^\dagger, B^\dagger]P_0)|^2 \leq (B, B)_0 \text{Tr}(P_0[C^\dagger, [H_0, C]]P_0)$$
■

Finally, we can formulate our basic inequality.

**Proposition 2.6.** (*“Bogolyubov inequality”*)

*Let  $B, C \in Op(\mathcal{S}_H)$  be such that  $B^\dagger(\mathcal{S}_H), C(\mathcal{S}_H), C^\dagger(\mathcal{S}_H) \subset \mathcal{S}_H$ . Then*

$$(10) \quad |\text{Tr}(P_0[C^\dagger, B^\dagger]P_0)|^2 \leq \left( \frac{1}{E_1 - E_0} \sum_{j=1}^{N_0} (\|Be_j\|^2 + \|B^\dagger e_j\|^2) \right) \text{Tr}(P_0[C^\dagger, [H_0, C]]P_0),$$

where  $(e_j)$  is any o.n. basis in  $P_0(\mathcal{H})$ .

*Proof:* We have the estimate:

$$\begin{aligned} (B, B)_0 &= \sum_{n>0} \frac{1}{E_n - E_0} (\rho_{n0} + \rho_{0n})(B, B) \leq \frac{1}{E_1 - E_0} \sum_{n>0} [\rho_{n0}(B, B) + \rho_{0n}(B, B)] \\ &\leq \frac{1}{E_1 - E_0} \sum_{n=0}^{\infty} [\rho_{n0}(B, B) + \rho_{0n}(B, B)] = \frac{1}{E_1 - E_0} \sum_{n=0}^{\infty} [\rho_{n0}(B, B) + \rho_{n0}(B^\dagger, B^\dagger)] \\ &= \frac{1}{E_1 - E_0} \sum_{j=1}^{N_0} (\|Be_j\|^2 + \|B^\dagger e_j\|^2), \end{aligned}$$

where the last equality follows from statement (4) of the Lemma 2.2. Inserting this estimate into the inequality in the Proposition 2.5 we get the result. ■

## 3. LACK OF ORDERING

**3.1. Definitions and basic properties of considered systems.** Let  $\Lambda$  be a finite subset of the simple cubic lattice in  $d$  dimensions:  $\Lambda \subset \mathbb{Z}^d$ . We assume that  $\Lambda$  is a (discrete) hypercube and that *the number of sites along every edge is even*; let us fix  $2N$  to be the length of the hypercube edge:

$$(11) \quad \Lambda_N := \{\mathbf{x} \in \mathbb{Z}^d : -N + 1 \leq x_i \leq N, i = 1, \dots, d\}$$

By  $|\Lambda_N|$  we denote the number of sites in  $\Lambda_N$  i.e.  $|\Lambda_N| = (2N)^d$ . With every site  $\mathbf{x} \in \Lambda_N$  we associate a separable, infinite dimensional Hilbert space  $\mathcal{H}_{\mathbf{x}}$  and the Hilbert space associated to the whole system is the tensor product  $\mathcal{H}_{\Lambda_N} := \bigotimes_{\mathbf{x} \in \Lambda_N} \mathcal{H}_{\mathbf{x}}$ . We will consider two systems: *planar and spherical rotors*.

*Planar rotors.* A position of a planar rotor at a site  $\mathbf{x}$  is given by  $\mathbf{n}_{\mathbf{x}}$  – a unit vector in  $\mathbb{R}^2$  or equivalently by  $\varphi_{\mathbf{x}} \in [0, 2\pi[$ ; thus  $\mathcal{H}_{\mathbf{x}} = L^2(S^1)$  and the total Hilbert space  $\mathcal{H}_{\Lambda_N} = L^2(S^1 \times \dots \times S^1)$  is the Hilbert space of square integrable functions on  $|\Lambda_N|$  dimensional torus  $M$ .

*Spherical rotors.* A position of a spherical rotor at a site  $\mathbf{x}$  is given by  $\mathbf{n}_{\mathbf{x}}$  – a unit vector in  $\mathbb{R}^3$  or equivalently by angles  $(\theta_{\mathbf{x}}, \varphi_{\mathbf{x}})$ ; this time  $\mathcal{H}_{\mathbf{x}} = L^2(S^2)$  and the total Hilbert space  $\mathcal{H}_{\Lambda_N} = L^2(S^2 \times \dots \times S^2)$ . In this case  $M$  is a product of  $|\Lambda_N|$  copies of  $S^2$ .

In both situations the hamiltonian of the system is a sum of three terms:  $T$  – kinetic energy,  $V$  – interaction of rotors on different sites and  $H_{int}$  – interaction with external field:

$$(12) \quad H := T + V + H_{int}$$

The kinetic energy operator is given by  $T = -\frac{1}{2I}\Delta$ , for some positive constant  $I$  (a moment of inertia), where  $\Delta$  is the Laplace operator for the obvious riemannian structures of a (flat) torus or a product of spheres;  $V$  and  $H_{int}$  are operators of multiplication by smooth functions. *Therefore  $H$  is an elliptic, second order differential operator on compact riemannian manifold  $M$ , formally selfadjoint on  $C^\infty(M)$  and bounded from below*. It is known ([22]) that:

- $H$  is essentially selfadjoint and extends (uniquely) to a bounded from below, selfadjoint operator (still denoted by  $H$ ) on  $L^2(M)$ ;
- the spectrum of  $H$  consists of isolated eigenvalues with finite multiplicities;
- eigenvectors of  $H$  may be represented by smooth functions.

Thus the spectral decomposition of  $H$  is of the form given in (5). As a space  $\mathcal{S}_H$  we take  $C^\infty(M)$ .

Our riemannian manifolds  $M$  are complete. In such a case it is known (see e.g. [23]) that the *heat semigroup*  $e^{t\Delta}$  is an integral operator with a *strictly positive* kernel (the heat kernel)  $K(t, x, y)$  on  $]0, \infty[ \times M \times M$ . Therefore  $e^{-tT}$  is *positivity improving* and the same (by the Trotter product formula) is true for  $e^{-tH}$ . Therefore *the ground state is unique* i.e. in the decomposition (5) we have  $\dim P_0(\mathcal{H}) = 1$  (see e.g. Chapt. XIII of [21] for details or [31] for a very brief exposition).

By

$$\omega_0(A) := \text{Tr}(P_0 A P_0)$$

we shall denote the expectation value in the ground state.

Let  $A, C \in Op(C^\infty(M))$  and assume that  $A, C, A^\dagger, C^\dagger$  preserve  $C^\infty(M)$ . Then inequalities (9) and (10) read:

$$(13) \quad |\omega_0([C^\dagger, A^\dagger])|^2 \leq (A, A)_0 \omega_0([C^\dagger, [H_0, C]])$$

$$(14) \quad |\omega_0([C^\dagger, A^\dagger])|^2 \leq \frac{\omega_0(A^\dagger A + A A^\dagger)}{E_1 - E_0} \omega_0([C^\dagger, [H_0, C]])$$

Let us specify in details our hamiltonians.

*Spherical Rotors.* The kinetic energy operator:

$$(15) \quad T := \frac{1}{2I} \sum_{\mathbf{x}} L_{\mathbf{x}}^2, \quad L_{\mathbf{x}}^2 = -(\partial_{\theta_{\mathbf{x}}}^2 + \cot \theta_{\mathbf{x}} \partial_{\theta_{\mathbf{x}}} + \frac{1}{\sin^2 \theta_{\mathbf{x}}} \partial_{\varphi_{\mathbf{x}}}^2),$$

where  $I > 0$  is the moment of inertia of rotors (we assume that all rotors have equal moments of inertia). The rotor-rotor interaction:

$$(16) \quad V := \sum_{\mathbf{x}, \mathbf{y}} J_{\mathbf{xy}} \sum_{m=-l}^l Y_l^m(\mathbf{n}_{\mathbf{x}}) \overline{Y_l^m(\mathbf{n}_{\mathbf{y}})} \quad J_{\mathbf{xy}} := J(|\mathbf{x} - \mathbf{y}|),$$

where  $Y_l^m$  are *spherical harmonics* (see Appendix for conventions we use). For the function  $J : [0, \infty[ \rightarrow [0, \infty[$  we assume:

$$(17) \quad \sum_{\mathbf{x} \in \mathbb{Z}^d} J(|\mathbf{x}|) |\mathbf{x}|^2 =: K_J < \infty, \quad J(0) = 0.$$

The term describing the interaction with *an external magnetic field*  $h \in \mathbb{R}$  is given by:

$$(18) \quad H_{int} := -h \sum_{\mathbf{x}} P_l(\cos \theta_{\mathbf{x}}),$$

where  $P_l$  denotes the Legendre Polynomial (see Appendix).

*Planar Rotors.* The kinetic energy operator:

$$(19) \quad T = -\frac{1}{2I} \sum_{\mathbf{x}} \frac{\partial^2}{\partial \varphi_{\mathbf{x}}^2},$$

where  $I > 0$  is the moment of inertia of rotor. The rotor-rotor interaction:

$$(20) \quad V = \sum_{\mathbf{x}, \mathbf{y}} J_{\mathbf{xy}} \mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}} = \sum_{\mathbf{xy}} J_{\mathbf{xy}} (\cos \varphi_{\mathbf{x}} \cos \varphi_{\mathbf{y}} + \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{y}}),$$

the assumptions about  $J_{\mathbf{xy}}$  are as for spherical rotors i.e. (17). The interaction term with an external field:

$$(21) \quad H_{int} = -h \sum_{\mathbf{x}} \cos \varphi_{\mathbf{x}}$$



**3.2. From Bogoliubov inequality to estimate for magnetization.** In further consideration we shall focus on *magnetization* of the system as a measure of LRO. As is well known, this quantity is defined by:

$$(22) \quad m_N(h) := \frac{1}{|\Lambda_N|} \omega_0 \left( \sum_{\mathbf{x}} P_l(\cos \theta_{\mathbf{x}}) \right),$$

for *spherical rotors*, and

$$(23) \quad m_N(h) := \frac{1}{|\Lambda_N|} \omega_0 \left( \sum_{\mathbf{x}} \cos \varphi_{\mathbf{x}} \right)$$

for *planar* ones. Let us remark that according to our assumptions (cf (18,21)) the ground state  $\omega_0$  depends on  $h$ .

We are going to use (13) to get an estimate for magnetization. Operator  $A$  which appear in (13) will be an operator of multiplication by smooth function (bounded, since our manifolds are compact) and operator  $C$  will be first order differential operator with smooth coefficients. Clearly they belong to  $Op(C^\infty(M))$  and preserve  $C^\infty(M)$  (as do their adjoints).

**3.2.1. Spherical Rotors.** Following the idea of ([7]) we define operators:

$$(24) \quad A_{\mathbf{k}} := - \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \cos \varphi_{\mathbf{x}} \sin \theta_{\mathbf{x}} P'_l(\cos \theta_{\mathbf{x}})$$

$$(25) \quad C_{\mathbf{k}}^\dagger := \sum_{\mathbf{x}} e^{-i\mathbf{k}\mathbf{x}} L_{\mathbf{x}}^+ = \sum_{\mathbf{x}} e^{-i\mathbf{k}\mathbf{x}} e^{i\varphi_{\mathbf{x}}} (\partial_{\theta_{\mathbf{x}}} + i \cot \theta_{\mathbf{x}} \partial_{\varphi_{\mathbf{x}}})$$

where  $\mathbf{k}$  take values in the first Brillouin zone, i.e.

$$(26) \quad k_j \in \left\{ -\frac{\pi(N-1)}{N}, \dots, \frac{\pi(N-1)}{N}, \pi \right\} \quad \text{for } j = 1, \dots, d,$$

Notice, that (24) implies  $A_{-\mathbf{k}} = A_{\mathbf{k}}^\dagger$  therefore  $(A_{-\mathbf{k}}, A_{-\mathbf{k}})_0 = (A_{\mathbf{k}}, A_{\mathbf{k}})_0$  by Prop.2.3. Writing the inequality (13) for  $C^\dagger = C_{\mathbf{k}}^\dagger$ ,  $A = A_{\mathbf{k}}$  and for  $C^\dagger = C_{-\mathbf{k}}^\dagger$ ,  $A = A_{-\mathbf{k}}$  and adding them, we obtain the inequality:

$$(27) \quad \left| \omega_0([C_{-\mathbf{k}}^\dagger, A_{-\mathbf{k}}]) \right|^2 + \left| \omega_0([C_{\mathbf{k}}^\dagger, A_{\mathbf{k}}]) \right|^2 \leq (A_{\mathbf{k}}, A_{\mathbf{k}})_0 \omega_0 \left( [[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}] \right).$$

By the straightforward computation we have:

$$\begin{aligned} [C_{\mathbf{k}}^\dagger, A_{\mathbf{k}}] &= - \sum_{\mathbf{xy}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left[ e^{i\varphi_{\mathbf{x}}} (\partial_{\theta_{\mathbf{x}}} + i \cot \theta_{\mathbf{x}} \partial_{\varphi_{\mathbf{x}}}), \cos \varphi_{\mathbf{y}} \sin \theta_{\mathbf{y}} P'_l(\cos \theta_{\mathbf{y}}) \right] = \\ &= \frac{1}{2} \sum_{\mathbf{x}} e^{2i\varphi_{\mathbf{x}}} \{ 2 \cos \theta_{\mathbf{x}} P'_l(\cos \theta_{\mathbf{x}}) - l(l+1) P_l(\cos \theta_{\mathbf{x}}) \} - \frac{l(l+1)}{2} \sum_{\mathbf{x}} P_l(\cos \theta_{\mathbf{x}}), \end{aligned}$$

The system consisting of spherical rotors possess the rotational symmetry i.e the hamiltonian (12) commutes with the unitary operator  $U_\varphi$  of rotation by an angle  $\varphi$  (in all sites simultaneously):

$$U_\varphi H U_\varphi^* = H,$$

where  $(U_\varphi f)(\varphi_{\mathbf{x}}, \theta_{\mathbf{x}}, \varphi_{\mathbf{y}}, \theta_{\mathbf{y}}, \dots) := f(\varphi_{\mathbf{x}} - \varphi, \theta_{\mathbf{x}}, \varphi_{\mathbf{y}} - \varphi, \theta_{\mathbf{y}}, \dots)$ .

In particular  $U_\varphi P_0 U_\varphi^* = P_0$ . Since  $U_\varphi e^{i\varphi_{\mathbf{x}}} U_\varphi^* = e^{i(\varphi_{\mathbf{x}} - \varphi)}$ ,

$$\begin{aligned} \omega_0(e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}})) &= \text{Tr}(P_0 e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}}) P_0) = \text{Tr}(U_\varphi P_0 e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}}) P_0 U_\varphi^*) = \text{Tr}(P_0 U_\varphi e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}}) U_\varphi^* P_0) = \\ &= e^{-i\varphi} \text{Tr}(P_0 e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}}) P_0) = e^{-i\varphi} \omega_0(e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}})), \end{aligned}$$

for any (continuous) function  $F$ . Therefore  $\omega_0(e^{i\varphi_{\mathbf{x}}} F(\theta_{\mathbf{x}})) = 0$ . In this way, using notation (22) we get :

$$\omega_0([C_{\mathbf{k}}^\dagger, A_{\mathbf{k}}]) = -\frac{l(l+1)}{2} \omega_0\left(\sum_{\mathbf{x}} P_l(\cos \theta_{\mathbf{x}})\right) = -\frac{l(l+1)}{2} |\Lambda_N| m_N(h) = \omega_0([C_{-\mathbf{k}}^\dagger, A_{-\mathbf{k}}]),$$

and the LHS of (27) can be expressed in the form:

$$(28) \quad 2 \left| \omega_0([C_{\mathbf{k}}^\dagger, A_{\mathbf{k}}]) \right|^2 = \frac{l^2(l+1)^2}{2} |\Lambda_N|^2 m_N(h)^2$$

Let us now compute the RHS of (27). We start with  $[C_{\mathbf{k}}^\dagger, T]$ :

$$[C_{\mathbf{k}}^\dagger, T] = \frac{1}{2I} \left[ \sum_{\mathbf{x}} e^{-i\mathbf{k}\mathbf{x}} L_{\mathbf{x}}^+, \sum_{\mathbf{y}} L_{\mathbf{y}}^2 \right] = \frac{1}{2I} \sum_{\mathbf{x}} e^{-i\mathbf{k}\mathbf{x}} [L_{\mathbf{x}}^+, L_{\mathbf{x}}^2] = 0$$

Notice that  $C_{\mathbf{k}}^\dagger$  is a first order differential operator and  $H_{int}$ ,  $V$  are operators of multiplication by smooth functions therefore  $[C_{\mathbf{k}}^\dagger, H_0] = [C_{\mathbf{k}}^\dagger, V + H_{int}]$  is an operator of multiplication by a smooth function as well, *so it is bounded* and the same is true for  $[[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}]$ . Let us calculate:

$$\begin{aligned} [[C_{\mathbf{k}}^\dagger, H_{int}], C_{\mathbf{k}}] &= \sum_{\mathbf{x}, \mathbf{y}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} [[L_{\mathbf{x}}^+, H_{int}], L_{\mathbf{y}}^-] = -h \sum_{\mathbf{x}, \mathbf{y}, \mathbf{z}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} [[L_{\mathbf{x}}^+, P_l(\cos \theta_{\mathbf{z}})], L_{\mathbf{y}}^-] = \\ &= -h \sum_{\mathbf{y}, \mathbf{z}} e^{-i\mathbf{k}(\mathbf{z}-\mathbf{y})} [-e^{i\varphi_{\mathbf{z}}} \sin \theta_{\mathbf{z}} P_l'(\cos \theta_{\mathbf{z}}), L_{\mathbf{y}}^-] = -h \sum_{\mathbf{y}} [L_{\mathbf{y}}^-, e^{i\varphi_{\mathbf{y}}} \sin \theta_{\mathbf{y}} P_l'(\cos \theta_{\mathbf{y}})] = \\ &= hl(l+1) \sum_{\mathbf{y}} P_l(\cos \theta_{\mathbf{y}}), \end{aligned}$$

where in the last equality, the Legendre equation (42) was used. It shows that the commutator does not depend on  $\mathbf{k}$  and in particular

$$(29) \quad [[C_{\mathbf{k}}^\dagger, H_{int}], C_{\mathbf{k}}] = hl(l+1) \sum_{\mathbf{x}} P_l(\cos \theta_{\mathbf{x}}) = [[C_{-\mathbf{k}}^\dagger, H_{int}], C_{-\mathbf{k}}].$$

Let us compute  $[[C_{\mathbf{k}}^\dagger, V], C_{\mathbf{k}}]$ .

$$[[C_{\mathbf{k}}^\dagger, V], C_{\mathbf{k}}] = \sum_{\mathbf{x}, \mathbf{t}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{t})} [[L_{\mathbf{x}}^+, V], L_{\mathbf{t}}^-] = \sum_{\mathbf{x}, \mathbf{t}, \mathbf{y}, \mathbf{z}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{t})} J_{\mathbf{y}\mathbf{z}} \sum_{m=-l}^l (-1)^m [[L_{\mathbf{x}}^+, Y_l^m(\mathbf{n}_{\mathbf{y}}) Y_l^{-m}(\mathbf{n}_{\mathbf{z}})], L_{\mathbf{t}}^-]$$

Now, using (43) we get:

$$\begin{aligned} [L_{\mathbf{x}}^+, Y_l^m(\mathbf{n}_{\mathbf{y}}) Y_l^{-m}(\mathbf{n}_{\mathbf{z}})] &= -\delta_{\mathbf{x}\mathbf{y}} \sqrt{(l-m)(l+m+1)} Y_l^{m+1}(\mathbf{n}_{\mathbf{y}}) Y_l^{-m}(\mathbf{n}_{\mathbf{z}}) + \\ &\quad -\delta_{\mathbf{x}\mathbf{z}} \sqrt{(l+m)(l-m+1)} Y_l^m(\mathbf{n}_{\mathbf{y}}) Y_l^{-m+1}(\mathbf{n}_{\mathbf{z}}) \end{aligned}$$

and by (43) again for  $[L_{\mathbf{t}}^-, Y_l^{m+1}(\mathbf{n}_y) Y_l^{-m}(\mathbf{n}_z)]$  we finally obtain:

$$[[C_{\mathbf{k}}^\dagger, V], C_{\mathbf{k}}] = \sum_{\mathbf{y}, \mathbf{z}} J_{\mathbf{yz}} \left( 2 \{ \cos \mathbf{k}(\mathbf{y} - \mathbf{z}) - 1 \} \sum_{m=-l}^l \{ l(l+1) - m^2 \} Y_l^m(\mathbf{n}_y) \overline{Y_l^m(\mathbf{n}_z)} + \right. \\ \left. - 2i \sin \mathbf{k}(\mathbf{y} - \mathbf{z}) \sum_{m=-l}^l m Y_l^m(\mathbf{n}_y) \overline{Y_l^m(\mathbf{n}_z)} \right).$$

Replacing  $\mathbf{k}$  by  $-\mathbf{k}$  in the formula above and adding both expressions we have

$$[[C_{\mathbf{k}}^\dagger, V], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, V], C_{-\mathbf{k}}] = 2 \sum_{\mathbf{yz}} J_{\mathbf{yz}} 2 \{ \cos \mathbf{k}(\mathbf{y} - \mathbf{z}) - 1 \} \sum_{m=-l}^l \{ l(l+1) - m^2 \} Y_l^m(\mathbf{n}_y) \overline{Y_l^m(\mathbf{n}_z)}$$

and by (29):

$$\omega_0 \left( [[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}] \right) = 2hl(l+1)|\Lambda_N| m_N(h) + \\ + 2 \sum_{\mathbf{yz}} J_{\mathbf{yz}} 2 \{ \cos \mathbf{k}(\mathbf{y} - \mathbf{z}) - 1 \} \sum_{m=-l}^l \{ l(l+1) - m^2 \} \omega_0 \left( Y_l^m(\mathbf{n}_y) \overline{Y_l^m(\mathbf{n}_z)} \right).$$

Notice that by (27):

$$\omega_0 \left( [[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}] \right) = \left| \omega_0 \left( [[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}] \right) \right|$$

and the estimation follows

$$\left| \omega_0 \left( [[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}] \right) \right| \leq 2|h|l(l+1)|\Lambda_N| |m_N(h)| + \\ (30) \quad + 2 \sum_{\mathbf{yz}} |J_{\mathbf{yz}}| 2 |\cos \mathbf{k}(\mathbf{y} - \mathbf{z}) - 1| \sum_{m=-l}^l \{ l(l+1) - m^2 \} \left| \omega_0 \left( Y_l^m(\mathbf{n}_y) \overline{Y_l^m(\mathbf{n}_z)} \right) \right|.$$

Let

$$\mathbf{Y}(l) := \max \{ \sup |Y_l^m(\mathbf{n})|, m = -l, \dots, l \}.$$

Then

$$\left| \omega_0 \left( Y_l^m(\mathbf{n}_y) \overline{Y_l^m(\mathbf{n}_z)} \right) \right| \leq \mathbf{Y}(l)^2.$$

Now, using the elementary inequality  $2|\cos \mathbf{k}(\mathbf{y} - \mathbf{z}) - 1| \leq |\mathbf{k}|^2 |\mathbf{y} - \mathbf{z}|^2$  and the formula  $\sum_{m=-l}^l m^2 = l(l+1)(2l+1)/3$  we can estimate the RHS of (30) by:

$$2|h|l(l+1)|\Lambda_N| |m_N(h)| + \frac{2}{3} \mathbf{Y}(l)^2 2l(l+1)(2l+1) |\mathbf{k}|^2 \sum_{\mathbf{yz}} |J_{\mathbf{yz}}| |\mathbf{y} - \mathbf{z}|^2$$

By assumption (17):  $\sum_{\mathbb{Z}^d} J(|\mathbf{x}|) |\mathbf{x}|^2 \leq K_J$  therefore  $\sum_{\mathbf{yz}} J_{\mathbf{yz}} |\mathbf{y} - \mathbf{z}|^2 \leq K_J |\Lambda_N|$  and we finally get:

$$\left| \omega_0 \left( [[C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}] + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}] \right) \right| \leq \\ (31) \quad 2l(l+1)|\Lambda_N| \left[ |h| |m_N(h)| + \frac{2}{3} (2l+1) \mathbf{Y}(l)^2 K_J |\mathbf{k}|^2 \right]$$

Due to this estimate and the formula (28) we write the inequality (27) as:

$$\frac{l^2(l+1)^2}{2} |\Lambda_N|^2 m_N(h)^2 \leq (A_{\mathbf{k}}, A_{\mathbf{k}})_0 2l(l+1) |\Lambda_N| \left[ |h| |m_N(h)| + \frac{2}{3} (2l+1) \mathbf{Y}(l)^2 K_J |\mathbf{k}|^2 \right]$$

or simply

$$\frac{l(l+1) |\Lambda_N| m_N(h)^2}{4} \times \frac{1}{|h| |m_N(h)| + \frac{2}{3} (2l+1) \mathbf{Y}(l)^2 K_J |\mathbf{k}|^2} \leq (A_{\mathbf{k}}, A_{\mathbf{k}})_0.$$

Summing over  $\mathbf{k}$ , where the range of summation is defined in (26), we get:

$$(32) \quad \frac{l(l+1) |\Lambda_N| m_N(h)^2}{4} \sum_{\mathbf{k}} \frac{1}{|h| |m_N(h)| + \frac{2}{3} (2l+1) \mathbf{Y}(l)^2 K_J |\mathbf{k}|^2} \leq \sum_{\mathbf{k}} (A_{\mathbf{k}}, A_{\mathbf{k}})_0.$$

Since  $|m_N(h)|$  can be estimated as:

$$|m_N(h)| = \frac{1}{|\Lambda_N|} \left| \omega_0 \left( \sum_{\mathbf{x}} P_l(\cos \theta_{\mathbf{x}}) \right) \right| \leq \frac{1}{|\Lambda_N|} \sum_{\mathbf{x}} |\omega_0(P_l(\cos \theta_{\mathbf{x}}))| \leq \sqrt{\frac{4\pi}{2l+1}} \mathbf{Y}(l)$$

we can replace  $|m_N(h)|$  in the denominator of (32) by  $\sqrt{\frac{4\pi}{2l+1}} \mathbf{Y}(l)$ :

$$\sqrt{\frac{2l+1}{4\pi}} \frac{l(l+1) |\Lambda_N| m_N(h)^2}{4 \mathbf{Y}(l)} \sum_{\mathbf{k}} \frac{1}{|h| + \frac{2}{3} (2l+1)^{3/2} (4\pi)^{-1/2} \mathbf{Y}(l) K_J |\mathbf{k}|^2} \leq \sum_{\mathbf{k}} (A_{\mathbf{k}}, A_{\mathbf{k}})_0,$$

Finally we can write our inequality as:

$$(33) \quad m_N(h)^2 \frac{(2\pi)^d}{|\Lambda_N|} \sum_{\mathbf{k}} \frac{1}{|h| + K^2 |\mathbf{k}|^2} \leq 4 (2\pi)^d \frac{\mathbf{Y}(l)}{l(l+1)} \sqrt{\frac{4\pi}{2l+1}} F_N(h),$$

where we have put  $K := \sqrt{\frac{2}{3} (2l+1)^{3/2} (4\pi)^{-1/2} \mathbf{Y}(l) K_J}$  (note that  $K$  does not depend on  $N$ ) and

$$(34) \quad F_N(h) := \frac{1}{|\Lambda_N|^2} \sum_{\mathbf{k}} (A_{\mathbf{k}}, A_{\mathbf{k}})_0$$

3.2.2. *Planar rotors.* This time we define operators:

$$(35) \quad C_{\mathbf{k}}^\dagger := \sum_{\mathbf{x}} e^{-i\mathbf{k}\mathbf{x}} \partial_{\varphi_{\mathbf{x}}} \quad , \quad A_{\mathbf{k}} := - \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \sin \varphi_{\mathbf{x}}$$

As for spherical rotors we calculate their commutator

$$[C_{\mathbf{k}}^\dagger, A_{\mathbf{k}}] = - \sum_{\mathbf{xy}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} [\partial_{\varphi_{\mathbf{x}}}, \sin \varphi_{\mathbf{y}}] = - \sum_{\mathbf{x}} \cos \varphi_{\mathbf{x}} = [C_{-\mathbf{k}}^\dagger, A_{-\mathbf{k}}].$$

and get the LHS of the inequality (27):

$$(36) \quad 2|\omega_0([C_{\mathbf{k}}^\dagger, A_{\mathbf{k}}])|^2 = 2|\Lambda_N|^2 m_N(h)^2,$$

where, in this case, the magnetization is defined by (23).

Now, for the RHS of (27):

$$[C_{\mathbf{k}}^\dagger, T] = 0$$

$$[[C_{\mathbf{k}}^\dagger, H_{int}], C_{\mathbf{k}}] = h \sum_{\mathbf{xyz}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{z})} [[\partial_{\varphi_{\mathbf{x}}}, \cos \varphi_{\mathbf{y}}], \partial_{\varphi_{\mathbf{z}}}] = h \sum_{\mathbf{xz}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{z})} [-\sin \varphi_{\mathbf{x}}, \partial_{\varphi_{\mathbf{z}}}] = h \sum_{\mathbf{x}} \cos \varphi_{\mathbf{x}}$$

$$[[C_{\mathbf{k}}^\dagger, V], C_{\mathbf{k}}] = \sum_{\mathbf{xyzt}} J_{\mathbf{yz}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{t})} [[\partial_{\varphi_{\mathbf{x}}}, \cos \varphi_{\mathbf{y}} \cos \varphi_{\mathbf{z}} + \sin \varphi_{\mathbf{y}} \sin \varphi_{\mathbf{z}}], \partial_{\varphi_{\mathbf{t}}}]$$

and, after routine calculations, we get:

$$\begin{aligned} [[C_{\mathbf{k}}^\dagger, V], C_{\mathbf{k}}] &= 2 \sum_{\mathbf{xy}} J_{\mathbf{xy}} (1 - e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}) (\cos \varphi_{\mathbf{x}} \cos \varphi_{\mathbf{y}} + \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{y}}) = \\ &= 2 \sum_{\mathbf{xy}} J_{\mathbf{xy}} (1 - \cos \mathbf{k}(\mathbf{x} - \mathbf{y})) (\cos \varphi_{\mathbf{x}} \cos \varphi_{\mathbf{y}} + \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{y}}) \end{aligned}$$

(we have used the symmetry  $J_{\mathbf{xy}} = J_{\mathbf{yx}}$ ). As for spherical rotors we obtain:

$$\begin{aligned} |\omega_0([C_{\mathbf{k}}^\dagger, H_0], C_{\mathbf{k}}) + [[C_{-\mathbf{k}}^\dagger, H_0], C_{-\mathbf{k}}])| &\leq 2|h||\Lambda_N| |m_N(h)| + \\ &4 \sum_{\mathbf{xy}} |J_{\mathbf{xy}}| |1 - \cos \mathbf{k}(\mathbf{x} - \mathbf{y})| |\omega_0(\cos \varphi_{\mathbf{x}} \cos \varphi_{\mathbf{y}} + \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{y}})| \end{aligned}$$

Now, using inequalities  $2|1 - \cos \mathbf{k}(\mathbf{x} - \mathbf{y})| \leq |\mathbf{k}|^2 |\mathbf{x} - \mathbf{y}|^2$ ,  $\sum_{\mathbf{xy}} J_{\mathbf{xy}} |\mathbf{x} - \mathbf{y}|^2 \leq K_J |\Lambda_N|$ , and  $|\omega_0(\cos \varphi_{\mathbf{x}} \cos \varphi_{\mathbf{y}} + \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{y}})| \leq 1$  we have

$$|\omega_0([C_{\mathbf{k}}^\dagger, H], C_{\mathbf{k}}) + [[C_{-\mathbf{k}}^\dagger, H], C_{-\mathbf{k}}])| \leq 2|\Lambda_N| (|h| |m_N(h)| + K_J |\mathbf{k}|^2);$$

and using (27) and (36):

$$2|\Lambda_N|^2 |m_N(h)|^2 \leq 2(A_{\mathbf{k}}, A_{\mathbf{k}})_0 |\Lambda_N| (|h| |m_N(h)| + K_J |\mathbf{k}|^2),$$

which, in turn (note the obvious estimate  $|m_N(h)| \leq 1$ ), can be written as:

$$m_N(h)^2 \frac{(2\pi)^d}{|\Lambda_N|} \times \frac{1}{|h| + K_J |\mathbf{k}|^2} \leq \frac{(2\pi)^d}{|\Lambda_N|^2} (A_{\mathbf{k}}, A_{\mathbf{k}})_0$$

Finally, performing summation over  $\mathbf{k}$  (as for spherical rotors) and using (34) :

$$(37) \quad m_N(h)^2 \frac{(2\pi)^d}{|\Lambda_N|} \sum_{\mathbf{k}} \frac{1}{|h| + K^2 |\mathbf{k}|^2} \leq (2\pi)^d F_N(h),$$

where, this time, we have put  $K := \sqrt{K_J}$ . This way we have the same expression (formally) as (33).

**3.3. Conditions for vanishing magnetization.** In dimensions  $d = 1$  or  $d = 2$ , the boundeness of the RHS of (37) (or (33)) forces magnetization to vanish (in  $h \rightarrow 0$ ) limit. More precisely :

**Proposition 3.1.** *Let  $d = 1$  or  $d = 2$ . If for some  $\delta > 0$ :*

$$\sup \{F_N(h) : 0 < |h| \leq \delta, N \in \mathbb{N}\} < \infty,$$

*then*

$$(38) \quad \lim_{h \rightarrow 0} \limsup_{N \rightarrow \infty} |m_N(h)| = 0.$$

*Proof:* Let us denote, for  $h \neq 0$ ,

$$S_N(h) := \frac{(2\pi)^d}{|\Lambda_N|} \sum_{\mathbf{k}} \frac{1}{|h| + K^2|\mathbf{k}|^2} > 0, \quad I_d(h) := \int_{[-\pi, \pi]^d} \frac{d\mathbf{k}}{|h| + K^2|\mathbf{k}|^2}.$$

Then it is easy to see that  $\lim_{N \rightarrow \infty} S_N(h) = I_d(h)$  and for  $d = 1, 2$   $\lim_{h \rightarrow 0} I_d(h) = \infty$ . Take  $h \neq 0$  such that  $|h| \leq \delta$ . By the assumption, for any  $N$  we can write inequalities (33) and (37) as

$$|m_N(h)| \leq \sqrt{\frac{M}{S_N(h)}}$$

for some positive constant  $M < \infty$ . Let us define (for  $h \neq 0$ ) a sequence  $s_N(h) := \inf\{S_{N'}(h), N' \geq N\}$ . Since  $\lim_{N \rightarrow \infty} s_N(h) = I_d(h)$ ,  $s_N(h) > 0$  for sufficiently large  $N$ ,  $N \geq N_0$ . Now, for  $N \geq N_0$ :

$$|m_N(h)| \leq \sqrt{\frac{M}{S_N(h)}} \leq \sqrt{\frac{M}{s_{N_0}(h)}}$$

therefore

$$\sup_{N \geq N_0} |m_N(h)| \leq \sqrt{\frac{M}{s_{N_0}(h)}}$$

and

$$\lim_{N_0 \rightarrow \infty} \sup_{N \geq N_0} |m_N(h)| \leq \sqrt{\frac{M}{I_d(h)}}$$

Now the assertion is clear. ■

**Remark.** The inequality (38) is weaker than the standard ones. For instance, in  $d = 1$  Heisenberg models, one obtains more explicit estimation [1]:

$$m(h) \leq \text{Const } h^{1/3} T^{-2/3}.$$

However, in such estimations, certain somewhat hidden fact is used when the thermodynamic limit is taken. Namely, the existence of free energy and magnetization in thermodynamic limit is assumed. It is in fact true [24]. However, we didn't use this fact and the estimation (38) is weaker than the standard ones, but sufficient to show the absence of the spontaneous magnetization at zero magnetic field.

We describe the sufficient condition to ensure the boundedness of  $F_N(h)$ . Let us *assume* the gap in energy spectrum:

$$\inf_h \inf_{\Lambda_N} \{E_1 - E_0\} =: \Delta > 0.$$

In this situation, the computation in the proof of Proposition 2.6 applied to  $A_{\mathbf{k}}$  leads to the estimate (since  $A_{\mathbf{k}}$  is normal):

$$(A_{\mathbf{k}}, A_{\mathbf{k}})_0 \leq \frac{2}{\Delta} \omega_0(A_{\mathbf{k}} A_{\mathbf{k}}^*)$$

Using the definition (24) of  $A_{\mathbf{k}}$  we compute for *spherical rotors*:

$$\begin{aligned} \sum_{\mathbf{k}} A_{\mathbf{k}}^* A_{\mathbf{k}} &= \sum_{\mathbf{k}} \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \cos \varphi_{\mathbf{x}} \sin \theta_{\mathbf{x}} P'_l(\cos \theta_{\mathbf{x}}) \cos \varphi_{\mathbf{y}} \sin \theta_{\mathbf{y}} P'_l(\cos \theta_{\mathbf{y}}) = \\ &= |\Lambda_N| \sum_{\mathbf{x}} \cos^2 \varphi_{\mathbf{x}} \sin^2 \theta_{\mathbf{x}} (P'_l(\cos \theta_{\mathbf{x}}))^2 \leq |\Lambda_N|^2 \tilde{Y}(l), \end{aligned}$$

where  $\tilde{Y}(l) := \sup \{(P'_l(\cos \theta))^2 \sin^2 \theta\}$  and we have used  $\sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} = |\Lambda_N| \delta(\mathbf{x} - \mathbf{y})$ . Thus  $F_N(h) \leq \tilde{Y}(l)$  and the upper bound does not depend on  $\Lambda_N$  and  $h$ .

For *planar rotors* we calculate:

$$\sum_{\mathbf{k}} A_{\mathbf{k}}^* A_{\mathbf{k}} = \sum_{\mathbf{k}} \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \sin \varphi_{\mathbf{x}} \sin \varphi_{\mathbf{y}} = |\Lambda_N| \sum_{\mathbf{x}} \sin^2 \varphi_{\mathbf{x}} \leq |\Lambda_N|^2$$

and, in this case,  $F_N(h) \leq 1$ .

### 3.4. Arguments for existence of energy gap in one-dimensional system of planar rotors.

In this subsection we argue that an energy gap should be present in the system of rotors if their coupling is sufficiently small. Consequently, such a system should not be ordered also in the ground state.

More precisely, we analyse one-dimensional system of planar rotors with nearest neighbour interactions. We slightly redefine (by a simple rescaling) the Hamiltonian written in the Subsec. 3.1:

$$(39) \quad H = T + V,$$

where

$$\begin{aligned} T &= - \sum_{\mathbf{x}} \frac{\partial^2}{\partial \varphi_{\mathbf{x}}^2}, \\ V &= \lambda \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \cos(\varphi_{\mathbf{x}} - \varphi_{\mathbf{y}}) - h \sum_{\mathbf{x}} \cos \varphi_{\mathbf{x}} \end{aligned}$$

The total number of rotors is  $N$ , and we impose periodic boundary condition. We will analyse energy levels of the system in the framework of perturbation theory, assuming that the coupling constant  $\lambda$  characterizing interaction between rotors is sufficiently small.

First, let us consider the rotor system in zero magnetic field, i.e. the case  $h = 0$ . The unperturbed system (i.e. for  $\lambda = 0$ ) is described by the Hamiltonian  $T$ ; this is a system of  $N$  non-interacting rotors. For a single rotor, the eigenfunctions  $\psi_k$  and corresponding eigenenergies  $E_k$  are

$$(40) \quad \psi_k = e^{ik\phi}, \quad E_k = k^2, \quad k \in \mathbb{Z}$$

Sometimes it is more convenient to use the real 'trigonometric' basis:

$$(41) \quad s_k = \sin k\phi, \quad k \in \mathbb{N}; \quad c_k = \cos k\phi, \quad k \in \mathbb{N} \cup \{0\}.$$

(eigenenergies of both  $c_k$  and  $s_k$  are  $E_k = k^2$ ). For  $N$  rotors, the eigenfunction is a tensor product of  $N$  eigenfunctions on every site:

$$\Psi \equiv \Psi_{k_1, \dots, k_N} = \psi_{1, k_1} \otimes \psi_{2, k_2} \otimes \dots \otimes \psi_{N, k_N}$$

(for  $\psi_{m, k_m}$ , the first index  $m$  is the site index, and  $k_m$  is the number of eigenfunction on the site  $m$ ). For trigonometric basis we have analogous expression. The eigenenergy of the  $\Psi$  function is

$$E_\Psi = k_1^2 + k_2^2 + \dots + k_N^2$$

The *unique* ground state  $\Psi_0$  corresponds to  $k_1 = k_2 = \dots = k_N = 0$  and energy  $E_0 = 0$ .

The first excited state is obtained when one of rotors (say,  $j$ -th) is in the first excited state (i.e.  $k_j = \pm 1$ ) whereas remaining rotors are in ground states. It is clear that the first excited state of the system has degeneracy equal to  $2N$ . Its eigenenergy  $E_1$  is:  $E_1 = 1$ . Notice that the energy gap  $E_1 - E_0 = 1$  and that *it is independent of  $N$* .

Now, let us calculate first-order correction to the ground state and the first excited state. We apply the standard perturbation theory, see for instance [25] or [21].

*The ground state.* The first-order correction  $\Delta E_0^{(1)}$  is zero:

$$\Delta E_0^{(1)} = \langle \Psi_0 | V | \Psi_0 \rangle = 0$$

*The first excited state.* The unperturbed state has  $2N$ -fold degeneracy, so the degenerate perturbation theory must be applied. Let us enumerate eigenfunctions of unperturbed system, which correspond to energy  $E_1$  as:  $\Psi_{1,1}, \Psi_{1,2}, \dots, \Psi_{1,j}, \dots, \Psi_{1,2N}$ . Define the  $V_{jm}$  matrix as:

$$V_{jm} = \langle \Psi_{1,j} | V | \Psi_{1,m} \rangle$$

The 1-st order corrections to the first excited state are eigenvalues of the  $V_{jm}$  matrix. We will apply the trigonometric basis (41). By a straightforward calculation, it turns out that macierz  $V_{jm}$  factorizes into two identical matrices. Due to cyclic boundary conditions, they are *cyclic* matrices:  $V_{j,j+1} = V_{j,j-1} = \lambda$ ,  $V_{1,N} = V_{N,1} = \lambda$ ; other matrix elements are zero. Their eigenvalues  $\lambda_n$  are:

$$\lambda_n = 2\lambda \cos \frac{2\pi n}{N}.$$

The lowest value among first-order correction is:

$$E_{1,min}^{(1)} = 1 - 2\lambda$$

and this is also the value of energy gap in the first order perturbation theory. Notice that this is *independent of  $N$* .

*Remark.* The reasoning above can be extended to the following situations:

- In considerations above we assumed that the magnetic field is zero. For non-zero magnetic fields, the unperturbed Hamiltonian corresponds to uncoupled rotors in periodic potential. Eigenvalues and eigenvectors of such a system are explicitly known and are expressible by *Mathieu function* [26]. They are much more involved, but the result is the same as in the case of zero magnetic field: *For sufficiently small magnetic field, the system of rotors*



*exhibits the energy gap above the ground state in the first order of perturbation theory. The value of gap is independent of  $N$ .*

- The interaction of rotors can be arbitrary finite range, translationally invariant but small; under these assumptions, the energy gap is also present in the first order of perturbation theory.
- These results hold also for *two-dimensional* lattice, by a straightforward extension of arguments above.

#### 4. SUMMARY, PERSPECTIVES

In the paper, we have formulated and proved the Bogolyubov inequality for operators at zero temperature. So far this inequality has been known for matrices, and we were able to extend it to operators. We have also applied this inequality to the system of interacting rotors. We have shown that if the dimension of the lattice is 1 or 2, the interaction decreases sufficiently fast with a distance (cf. (17)) and there is an energy gap over the ground state, then the spontaneous magnetization in the ground state is zero, i.e. there is no LRO in the system.

We present also heuristic arguments suggesting that one- and two-dimensional system of interacting rotors has the energy gap independent of the system size if the interaction is sufficiently small. This would imply the lack of ordering in the ground state of such rotor system. The argument is based on perturbation theory. Unfortunately we were not able to proceed further with perturbation theory – the calculations become hopelessly complicated in further orders of perturbation theory. This way, the rigorous proof of disorder in the ground state of weakly interacting rotors is still an open problem. Perhaps other methods could be more adequate. One of them is *stochastic analysis* used in the anharmonic crystal model problems [27], [28]. Another potentially applicable method could be the rigorous renormalization group [29]. In the case where qualitative properties of unperturbed system are suspected to be not changed under switching on the perturbation, one can hope to show rigorously the convergence of RG perturbation series.

To conclude, let us mention some other open (to our best knowledge) problems, which seem for us to be very interesting.

For *large* couplings between rotors, the ground state is ordered (for  $d = 2$  and nearest-neighbour ferromagnetic couplings) [30], [31]. For *small* couplings, presumably there is no ordering in the ground state. So, for some intermediate value of coupling, we conjecture that the *quantum critical point* should appear. It would be very interesting to verify such a conjecture, and if it is true, to examine the nature of this critical point.

#### 5. APPENDIX

Here we collect definitions and formulae for spherical harmonics, Legendres' polynomials and angular momentum operators we use:

$$P_l(x) := \frac{(-1)^l}{2^l l!} \frac{d^l}{dx^l} (1 - x^2)^l$$

$$\begin{aligned}
(42) \quad & (1 - \cos^2 \theta) P_l''(\cos \theta) - 2 \cos \theta P_l'(\cos \theta) + l(l+1) P_l(\cos \theta) = 0; \\
& P_l^m(x) := (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad P_l^{-m}(x) := (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x), \quad m = 0, 1, \dots, l \\
& Y_l^m(\theta, \varphi) := (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta) e^{im\varphi} =: C_l^m P_l^{|m|}(\cos \theta) e^{im\varphi}, \quad m = -l, \dots, l \\
& C_l^m := \begin{cases} (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} & m \geq 0 \\ \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} & m < 0 \end{cases} \\
& \overline{Y_l^m} = (-1)^m Y_l^{-m} \\
& L^+ := e^{i\varphi} (\partial_\theta + i \cot \theta \partial_\varphi) \quad , \quad L^- := (L^+)^* = e^{-i\varphi} (-\partial_\theta + i \cot \theta \partial_\varphi) \quad , \quad L^z := -i \partial_\varphi \\
& [L^+, L^-] = 2L^z \quad , \quad [L^z, L^+] = L^+ \quad , \quad [L^z, L^-] = -L^- \\
& L^2 = -(\partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2) = L^+ L^- + (L^z)^2 - L^z \\
(43) \quad & [L^+, Y_l^m] = -\sqrt{(l-m)(l+m+1)} Y_l^{m+1}, \quad [L^-, Y_l^m] = -\sqrt{(l+m)(l-m+1)} Y_l^{m-1}
\end{aligned}$$

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